

# Lion and Man Game in Compact Spaces

Yufereva Olga  
yufereva12@gmail.com

**Abstract** The pursuit-evasion game with two persons is considered. Both players are moving in a metric space, have equal maximum speeds and complete information about the location of each other. We study the sufficient conditions for a capture (with positive capture radius). We assume that Lion wins if he manages the capture independently of the initial positions of the players and the evader's strategy. We prove that in lion-and-man game in a compact metric space  $(K, d)$  the simple pursuit strategy guarantees the win of Lion if each pair of points in  $K$  has a unique geodesic segment connecting them (in other words, if  $K$  is a unique geodesic space). We also do not need to use such properties as convexity, finite dimension, boundary regularity or contractibility of the loops. Examples of the spaces with successful pursuit strategy are considered, in particular, we investigate compact  $CAT(0)$ -spaces and compact subsets of  $\mathbb{R}^n$  with  $l_p$ -norms.

**Keywords** pursuit-evasion game · lion-and-man game · simple pursuit · unique geodesic space

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## 1 Introduction

In this article we study a pursuit-evasion game known as the lion-and-man game. Both players have equal opportunities and opposite goals. Namely Lion wants to catch Man, and Man wants to evade. There are many papers on pursuit-evasion games; see [Isaacs(1999), Petrosjan(1993), Pontryagin(1966), Chernous'ko(1976), Ivanov and Ledyaev(1983)] for example.

In the classical case the phase space is a closed disk in  $\mathbb{R}^2$ . Besicovitch showed that the evader can escape the pursuer infinitely long [Littlewood(1986)]. But the pursuer can achieve any positive distance to the evader [Littlewood(1986), Alonso et al(1992)]. Due to this, we say the pursuer wins if he can reduce the distance between the players down to any positive  $\varepsilon$ . The simple pursuit strategy implies that Lion wins in this classical case and in this article we use a simple pursuit too.

As regards the applications, lion-and-man game is used, e.g., by [Bramson et al(2014)] to study Brownian motions; there is a relation between the pursuit-evasion game and the space's metric properties such as contractibility of loops. A lot of examples of subsets of  $\mathbb{R}^n$  are also presented, metric properties and the result of the lion-and-man game are described. Results like [Isler and Noori(2015)] may benefit the robotics community. Similar papers [Isler and Karnad(2009), Tovar and LaValle(2008), O'Kane and Stiffler(2012)] devoted to this area differ in the visibility of players or capabilities of their motions.

One specific pursuer's strategy on convex terrains is described in [Isler and Noori(2015)]. Another Lion's strategy was introduced in [Sgall(2001)] to consider a pursuit within the non-negative quadrant of the plane. This was a discrete-time version that later was generalized for CAT(0)-spaces in [Beveridge and Cai(2015)] (CAT(0)-space is an Alexandrov space with non-positive curvature, see more in [Bridson and Haefliger(2011)]). Here is an important result of [Alexander et al(2010)]: the simple pursuit leads to capture in compact CAT(0)-spaces; it became a classical paper. In [Bramson et al(2014)], this result was expanded on finite-dimension CAT( $\kappa$ )-spaces of sufficiently small diameter.

In contrast with the above-mentioned articles, we show that the uniqueness of geodesic paths directly implies Lion's win on a compact if Lion conducts a simple pursuit. No convexity, finite dimension, regularity on the boundary, well-contractible loops, CAT(0)-spaces or CAT( $\kappa$ )-spaces are required. On the other hand, our result implies the result [Alexander et al(2010)] for compact CAT(0)-spaces, the result of [Bramson et al(2014), Theorem 4.6] for finite-dimension CAT( $\kappa$ )-spaces of sufficiently small diameter. Note that we want Lion to win independently of the Man's strategy and initial players' positions. In this way we can assume that Man knows the Lion's strategy in advance and can calculate their positions at an arbitrary time. We analyse some possible players' trajectories and their limit properties. So, we need the compactness to consider such a limit point.

This paper is organized as follows. In Section 2 we present simple examples of the game and exhibit the properties related to the Lion's successful strategy. Section 3 is devoted to properties of geodesic segments. In Section 4 we present the constraints on a metric space, dynamics and the Lion's strategy; and formulate the main result (Theorem 1). The whole Section 5 is devoted to the proof of this theorem.

## 2 Simple Examples

Consider the games corresponding to the following dynamics of players:

$$\begin{aligned} \text{Lion :} \quad & \dot{x} = u, & x(0) &= x^0, \\ \text{Man :} \quad & \dot{y} = v, & y(0) &= y^0, \\ & x, y \in K \subset \mathbb{R}^n, \\ & u, v \in Q_p \subset \mathbb{R}^n, \end{aligned} \tag{1}$$

where the set  $K$  is closed,  $p > 1$ ,

$$Q_p = \left\{ (a_1, a_2, \dots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^n |a_i|^p \leq 1 \right\}.$$

Lion uses the simple pursuit strategy. Lion wins if at time  $t^* \in [0, +\infty)$  the distance between  $x(t^*)$  and  $y(t^*)$  is less than  $\varepsilon$ . Man wins if such  $t^*$  does not exist.

Consider the game modelling (1) with explicit  $K$  and  $Q_p$ .

*Example 1* Let  $K$  be  $\mathbb{R}^2$ ,  $p = 2$ .

It is easy to see that Man wins by moving in the direction away from the Lion's initial position. A similar strategy leads to the Man's win in a lot of unbounded sets  $K$ . Though an example of capture on an unbounded space is reported [Bačák(2012)], we will later assume the compactness of  $K$ , as in [Alexander et al(2010), Bramson et al(2014)].

*Example 2* Let  $K$  be  $S^1 \subset \mathbb{R}^2$ ,  $p = 2$ .

There is no capture if Man goes towards the point opposite to the Lion's position. Since we always assume that Man knows Lion's position and actions, the condition of uniqueness of geodesic segments is essential for capture.

*Example 3*  $K$  is  $B^1 \subset \mathbb{R}^n$ ,  $p = 2$ .

In particular, [Alexander et al(2010)] proves that the continuous simple pursuit strategy leads to Lion's win because here  $K$  is a CAT(0)-space. We prove the analogous result for the discrete pursuit strategy.

In the examples below,  $K$  is not a CAT(0)-space and [Alexander et al(2010)] can not be employed.

*Example 4*  $K = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, 1 > z \geq c\}$ ,  $p = 2$ ,  $0 < c < 1$ .

It is a compact unique geodesic space hence Lion wins (by Theorem 1). It also follows from [Bramson et al(2014), Theorem 4.6] for finite-dimensional Alexandrov space with positive curvature.

*Example 5*  $K$  is  $B^1 \subset \mathbb{R}^n$ ,  $1 < p < \infty$ .

Unlike Example 3, here  $K$  is not an Alexandrov space if  $p \neq 2$ . But it is a unique geodesic space (see [Bridson and Haefliger(2011), Proposition I.4.16]). So Lion wins (see Theorem 1).

### 3 Geodesic Segments

We need to introduce new terms before describing Lion's strategy. Let  $(K, d)$  be a metric space. Definitions below are taken from [Bridson and Haefliger(2011)].

- A *geodesic path* joining  $A$  to  $B$  ( $A, B \in K$ ) is a map  $\gamma$  from a closed interval  $[a, b]$  to  $K$  such that  $\gamma(a) = A$ ,  $\gamma(b) = B$  and  $d(\gamma(t), \gamma(t')) = |t - t'|$  for all  $t, t' \in [a, b]$ .
- The image of  $\gamma$  is called a *geodesic segment* with the endpoints  $A$  and  $B$  and is denoted  $[AB]$ . And then  $B \in [AC]$  means that the point  $B$  belongs to the geodesic segment between  $A$  and  $C$ .
- A metric space  $K$  is called a *geodesic space* if every pair of points  $A, B \in K$  can be joined by a geodesic path.
- A metric space  $K$  is called a *uniquely geodesic space* if every pair of points  $A, B \in K$  can be joined by a unique geodesic path.

- Let  $K$  be a uniquely geodesic space. Denote by  $c(x, y)$  the geodesic segment joining  $x$  to  $y$ , which is the natural parameter function from  $[0, 1]$  to  $K$ . Geodesics in  $K$  are said to *vary continuously with their endpoints* if  $c(x_n, y_n)$  uniformly converges to  $c(x, y)$  as  $x_n \rightarrow x$  and  $y_n \rightarrow y$ .

All the following remarks given in this section hold true in a unique geodesic space.

*Remark 1*  $B \in [AC]$  is equivalent to  $[AB] \subset [AC]$ .

*Remark 2*  $B \in [AC]$  is equivalent to  $d(A, C) = d(A, B) + d(B, C)$ , where  $d$  is the metric of the considered space.

Moreover, we take the following lemma from [Bridson and Haefliger(2011), Corollary I.3.13].

**Lemma 1** *If a proper metric space  $K$  is uniquely geodesic, then geodesics in  $K$  vary continuously with their endpoints.*

It is used in the lemma below.

**Lemma 2** *A sequence of the same-length geodesic segments  $[A_n, B_n]$  has a limit point that is a geodesic segment.*

*Proof* All geodesic segments are parametrized by the arc length hence they are equicontinuous. Since  $K$  is compact, they are uniformly bounded. So, thanks to Arzela–Ascoli theorem, the considered sequence has a subsequence that converges to a continuous map. And this limit map is a geodesic path according to Lemma 1.

## 4 General Statement

### 4.1 Dynamics

Recall that players move in a metric space  $(K, d)$ . As we saw in Section 2, properties of  $K$  define the answer to the question which player wins. We are interested in conditions on  $K$  sufficient for the capture, including non-Euclidean  $K$ .

Assume that  $K$  is a unique geodesic space. For convenience, also set

$$\rho(A, B) = \max\{d(A_1, B_1), d(A_2, B_2)\}$$

for all  $A = (A_1, A_2), B = (B_1, B_2) \in K^2$ .

Let  $\mathcal{L}(\cdot), \mathcal{M}(\cdot)$  be maps from  $[0, \infty)$  to  $K$  such that  $\mathcal{L}(t), \mathcal{M}(t)$  denote the positions of Lion and Man at time  $t$ . Because players' speeds must not exceed one, we impose the following constraints:

$$d(\mathcal{L}(t_1), \mathcal{L}(t_2)) \leq |t_1 - t_2|, \quad d(\mathcal{M}(t_1), \mathcal{M}(t_2)) \leq |t_1 - t_2| \quad \forall t_1, t_2 \geq 0.$$

Lion wins if he can reduce the distance between him and Man down to any positive  $\varepsilon$ . Man can choose his trajectory arbitrarily among those that satisfy

the constraints above. Moreover, we assume that he knows Lion's strategy and their initial positions.

Let us fix this number  $\varepsilon$  and introduce the strategy that allows Lion to achieve the distance between the players that does not exceed  $\varepsilon$ .

#### 4.2 Description of the Lion's Strategy

The background of the chosen strategy is the desire to use a simple strategy that can be realized on a large class of spaces.

Let us define the Lion's strategy. Consider the sequence  $\Delta = \{\tau_i\}_{i=1}^\infty = \{i\varepsilon\}_{i=1}^\infty$ . At the corresponding intervals, Lion takes aim at Man and takes a step toward him; more exactly,

**Definition 1** A curve  $\zeta(\cdot) = (\mathcal{L}(\cdot), \mathcal{M}(\cdot))$ , from  $\mathbb{R}_+$  to  $K^2$ , where  $\mathcal{L}(\cdot)$  and  $\mathcal{M}(\cdot)$  are 1-Lipschitz continuous, is called the simple pursuit curve iff for each  $i \in \mathbb{N} \cup \{0\}$  the condition  $d(\mathcal{L}(\tau_i), \mathcal{M}(\tau_i)) \geq \varepsilon$  implies

$$d(\mathcal{L}(\tau_i), \mathcal{L}(\tau_{i+1})) = \varepsilon, \quad [\mathcal{L}(\tau_i), \mathcal{L}(\tau_{i+1})] \subset [\mathcal{L}(\tau_i), \mathcal{M}(\tau_i)].$$

So,  $\Delta$  is the sequence of control correction times.

Formally, we should consider the case  $d(\mathcal{L}(\tau_i), \mathcal{M}(\tau_i)) < \varepsilon$ . In this case, Lion can move along  $[\mathcal{L}(\tau_i), \mathcal{M}(\tau_i)]$  with speed one and stand at the point  $\mathcal{M}(\tau_i)$ , till time  $\tau_{i+1}$  if necessary.

Thus, this strategy can be realised because  $K$  is a unique geodesic space.

Note that while the distance between the players is not less than  $\varepsilon$ , the Lion's trajectory consists of geodesic segments:  $\mathcal{L}(\cdot)$  is 1-Lipschitz continuous hence the equality  $d(\mathcal{L}(\tau_i), \mathcal{L}(\tau_{i+1})) = \varepsilon$  implies that  $\mathcal{L}(\cdot)|_{[\tau_i, \tau_{i+1}]}$  is a geodesic path.

Studying the game with this strategy, we obtained the following result:

**Theorem 1** If  $(K, d)$  is a compact unique geodesic space, then the simple pursuit strategy guarantees that Lion wins.

### 5 Proof of Theorem 1

Let us show that Lion gets a capture. Suppose the contrary. Namely, we suppose that there exists a positive number  $\varepsilon$  and a simple pursuit curve  $\zeta(\cdot) = (\mathcal{L}(\cdot), \mathcal{M}(\cdot))$  such that Man will never be captured, i.e.,  $d(\mathcal{L}(t), \mathcal{M}(t)) > \varepsilon$  for all  $t \geq 0$ .

We will show that this assumption implies a contradiction.

#### 5.1 Good Curves

By our definition,  $\zeta(\cdot) = (\mathcal{L}(\cdot), \mathcal{M}(\cdot))$  is the curve along which the players will go. But we want to consider segments of simple pursuit curves along which the players could move if initial positions were different or Man took another strategy. Let us formulate it rigorously.

**Definition 2** For all  $\tau_a, \tau_b \in \Delta \cup \{\infty\}$ ,  $\tau_a < \tau_b$ , a curve  $\gamma(\cdot) = (\gamma_L(\cdot), \gamma_M(\cdot)) : [\tau_a, \tau_b] \rightarrow K^2$  is called a good curve iff

1.  $\gamma_L(\cdot), \gamma_M(\cdot)$  are 1-Lipschitz continuous,
2.  $d(\gamma_L(\tau_i), \gamma_L(\tau_{i+1})) = \varepsilon$   $\tau_a \leq \tau_i < \tau_b$ ,
3.  $[\gamma_L(\tau_i), \gamma_L(\tau_{i+1})] \subset [\gamma_L(\tau_i), \gamma_M(\tau_i)]$   $\tau_a \leq \tau_i < \tau_b$ ,
4.  $d(\gamma_L(\tau_i), \gamma_M(\tau_i)) \geq \varepsilon$   $\tau_a \leq \tau_i \leq \tau_b$ .

So, all statements that hold for good curves hold for  $\zeta(\cdot)$  too.

**Lemma 3** A sequence of good curves

$$\psi^n(\cdot) = (\psi_L^n(\cdot), \psi_M^n(\cdot)) : [0, \tau_i] \rightarrow K^2$$

has a subsequence that converges to a good curve

$$\psi(\cdot) = (\psi_L(\cdot), \psi_M(\cdot)) : [0, \tau_i] \rightarrow K^2.$$

*Proof* Since each  $\psi_L^n(\cdot)$  and  $\psi_M^n(\cdot)$  are 1-Lipschitz continuous, each  $\psi^n(\cdot)$  is 1-Lipschitz continuous too (by the definition of the metric  $\rho$ ). In addition,  $K^2$  is compact hence all of  $\psi^n(\cdot)$  are both equicontinuous and uniformly bounded. So this sequence has a limit point by Arzela–Ascoli theorem. Note that  $\psi_L(\cdot)$  and  $\psi_M(\cdot)$  are 1-Lipschitz, as well as each of  $\psi_L^n(\cdot)$  and  $\psi_M^n(\cdot)$ . In the same way we get Item 4 of good curves' definition. The remaining items follow from Lemma 2. Thus  $\psi(\cdot)$  is a good curve indeed.

## 5.2 Behaviour of Distance between Players

In this section we illustrate properties of good curves. Let us denote by  $d_\gamma(t)$  the distance between the components of a good curve  $\gamma(\cdot)$  at time  $t$ .

Proposition 1 uses the idea of the triangle inequality from [Alexander et al(2010)].

**Proposition 1** Let  $\gamma(\cdot) = (L(\cdot), M(\cdot))$  be a good curve,  $[\tau_i, \tau_{i+1}] \subset \text{dom}(\gamma)$ . The following statements are equivalent:

1.  $\forall t \in [\tau_i, \tau_{i+1}] \ d_\gamma(t) = d_\gamma(\tau_i)$ ,
2.  $d_\gamma(\tau_i) = d_\gamma(\tau_{i+1})$ ,
3.  $M(\tau_i) \in [L(\tau_i), M(\tau_{i+1})]$  and  $d(M(\tau_i), M(\tau_{i+1})) = \varepsilon$ .

*Proof* The Statement 2 trivially follows from Statement 1. Statement 1 follows from Statement 3 and Remark 2.

Let us assume Statement 2 and show that Statement 3 holds. Note that, by the triangle inequality and the definition of the good curve, we obtain

$$\begin{aligned} d_\gamma(\tau_{i+1}) &= d(L(\tau_{i+1}), M(\tau_{i+1})) \\ &\leq d(L(\tau_{i+1}), M(\tau_i)) + d(M(\tau_i), M(\tau_{i+1})) \leq d(L(\tau_{i+1}), M(\tau_i)) + \varepsilon \\ &= d(L(\tau_{i+1}), M(\tau_i)) + d(L(\tau_i), L(\tau_{i+1})) = d(L(\tau_i), M(\tau_{i+1})) = d_\gamma(\tau_i). \end{aligned}$$

Due to the equality from Statement 2, both obtained non-equality signs should be changed to equality ones, but it is possible only if both the conditions of Statement 3 hold.

There are also two remarks on such  $\gamma(\cdot)$ , which are a direct corollary of this proposition. We have two remarks held for such  $\gamma(\cdot)$ , which follow directly from this.

*Remark 3*  $d_\gamma(\cdot)$  does not increase.

*Remark 4* If  $d_\gamma(\tau_i) = d_\gamma(\tau_{i+1})$ , then the images of  $[\tau_i, \tau_{i+1}]$  under the map  $M(\cdot)$  coincide with the geodesic segments  $[M(\tau_i), M(\tau_{i+1})]$  and  $d(M(\tau_i), M(\tau_{i+1})) = \varepsilon$ .

And combining this proposition with the definition of the Lion's strategy, we get

*Remark 5* If  $d_\gamma(\tau_i) = d_\gamma(\tau_{i+1})$  then  $[L(\tau_i), M(\tau_i)] \subset [L(\tau_i), M(\tau_{i+1})]$ .

**Proposition 2** Let  $\gamma(\cdot) = (L(\cdot), M(\cdot))$  be a good curve such that  $[0, \tau_n] \subset \text{dom}(\gamma)$  and  $d_\gamma(0) = d_\gamma(\tau_n)$ ; then,  $L(\cdot)$  on  $[0, \tau_n]$  is a geodesic path.

*Proof* By virtue of the definition of  $L(\cdot)$ , more precisely, since  $d(L(\tau_i), L(\tau_{i+1})) = \varepsilon$ , the restriction of  $L(\cdot)$  to  $[0, \tau_n]$  is a geodesic path if and only if

$$\begin{aligned} L(0), L(\tau_1), \dots, L(\tau_n) &\in [L(0), L(\tau_n)], \\ [L(\tau_{n-1}), L(\tau_n)] &\subset [L(\tau_{n-2}), L(\tau_n)] \subset \dots \subset [L(0), L(\tau_n)]. \end{aligned}$$

But we will prove the more strong statements, namely, for all  $\tau_i < \tau_n$ ,

$$\begin{aligned} d(M(\tau_i), M(\tau_{i+1})) &= \varepsilon, \\ [L(\tau_n), M(\tau_n)] &\subset [L(\tau_{n-1}), M(\tau_n)] \subset \dots \subset [L(0), M(\tau_n)], \\ [L(0), M(0)] &\subset [L(0), M(\tau_1)] \subset \dots \subset [L(0), M(\tau_n)]. \end{aligned}$$

Notice that this involves the fact that the images of  $[0, \tau_n]$  under  $L(\cdot)$  and  $M(\cdot)$  are geodesic segments with the length  $n\varepsilon$ ; hence,  $L(\cdot)|_{[0, \tau_n]}$  is a geodesic path indeed.

If  $n = 1$ , the proof trivially follows from the definition of the Lion's strategy and Proposition 1. It is a basis of induction.

Let us assume that for some  $k$  ( $n = k$ ) these inclusions hold. Prove the step of the induction—the case  $n = k + 1$ .

Define a curve  $\eta(\cdot) = (\eta_L(\cdot), \eta_M(\cdot)) : [0, k\varepsilon] \rightarrow K^2$  as follows:

$$\eta_L(t) = L(t), \quad \eta_M(t) = M(t + \tau_1) \quad \forall t \in [0, k\varepsilon].$$

It is easy to see that  $\eta(\cdot)$  satisfies Items 1 and 2 from the definition of the good curve. Consider Item 3. The definition of the Lion's strategy and Remark 5 give, for all  $m$ ,  $0 \leq m < k$ ,

$$\begin{aligned} [\eta_L(m\varepsilon), \eta_L((m+1)\varepsilon)] &= [L(m\varepsilon), L((m+1)\varepsilon)] \\ &\subset [L(m\varepsilon), M(m\varepsilon)] \subset [L(m\varepsilon), M((m+1)\varepsilon)] \\ &= [\eta_L(m\varepsilon), \eta_M(m\varepsilon)]. \end{aligned}$$

Hence, Item 3 of this definition holds too. Let us check the last item and, at the same time, show the equality  $d_\eta(0) = d_\eta(k\varepsilon)$ . From the inductive hypothesis, we directly have that  $L|_{[\tau_0, \tau_k]}$  and  $M|_{[\tau_0, \tau_k]}$  are geodesic paths, hence, for all natural  $i$ ,  $0 \leq i < k$ ,

$$\begin{aligned} d_\eta(0) &= d(L(0), M(\tau_1)) \\ &= d(L(\tau_1), M(\tau_1)) + \varepsilon \\ &= d_\gamma(\tau_1) + \varepsilon \\ &= d_\gamma(\tau_i) + \varepsilon = d_\eta(i\varepsilon); \end{aligned}$$

moreover, the restriction  $\gamma|_{[\tau_1, \tau_{k+1}]}$  satisfies the induction hypothesis for the case  $n = k$ , hence  $L|_{[\tau_1, \tau_{k+1}]}$  and  $M|_{[\tau_1, \tau_{k+1}]}$  are geodesic paths too; that implies

$$\begin{aligned} d_\eta((k-1)\varepsilon) &= d(L(\tau_{k-1}), M(\tau_k)) = d(L(\tau_{k-1}), M(\tau_{k-1})) + \varepsilon \\ &= d(L(\tau_k), M(\tau_k)) + \varepsilon = d(L(\tau_k), M(\tau_{k+1})) = d_\eta(k\varepsilon). \end{aligned}$$

Thus,  $\eta(\cdot)$  is a good curve.

So, the condition  $d_\eta(0) = d_\eta(k\varepsilon)$  allows us to use induction hypothesis for the case  $n = k$  to curve  $\eta(\cdot)$ . In this way, we get

$$\begin{aligned} [\eta_L(k\varepsilon), \eta_M(k\varepsilon)] &\subset [\eta_L((k-1)\varepsilon), \eta_M(k\varepsilon)] \subset \dots \subset [\eta_L(0), \eta_M(k\varepsilon)], \\ [\eta_L(0), \eta_M(0)] &\subset [\eta_L(0), \eta_M(\varepsilon)] \subset \dots \subset [\eta_L(0), \eta_M(k\varepsilon)]. \end{aligned}$$

Substituting  $\gamma(\cdot)$  for  $\eta(\cdot)$ , we obtain

$$\begin{aligned} [L(\tau_k), M(\tau_{k+1})] &\subset [L(\tau_{k-1}), M(\tau_{k+1})] \subset \dots \subset [L(0), M(\tau_{k+1})], \\ [L(0), M(\tau_1)] &\subset [L(0), M(\tau_2)] \subset \dots \subset [L(0), M(\tau_{k+1})]. \end{aligned}$$

The remaining inclusions  $L(\tau_{k+1}) \in [L(\tau_k), M(\tau_{k+1})]$  and  $M(0) \in [L(0), M(\tau_1)]$  follow from the definition of the Lion's strategy and Proposition 1, respectively. The equality  $d(M(\tau_i), M(\tau_{i+1})) = \varepsilon$  for  $\tau_i$ ,  $\tau_0 \leq \tau_i < \tau_n$ , is given by  $\gamma|_{[\tau_0, \tau_k]}$  and  $\gamma|_{[\tau_1, \tau_{k+1}]}$  that satisfy induction hypothesis for the case  $n = k$ . So, we get what we need.

Thus, we proved this proposition for all natural  $n$ .

### 5.3 Rounds

We shall add the following useful construction.

**Definition 3** *The restriction of  $\zeta(\cdot)$  to an interval  $[\tau_i, \tau_j]$  is called a round for a set  $A \subset K^2$  iff*

1.  $\tau_i, \tau_j \in \Delta$ ;
2.  $\tau_j - \tau_i > \varepsilon$ , i.e.  $j - i > 1$ ;
3.  $\zeta(\tau_i) \in A$ ;
4.  $\zeta(\tau_j) \in A$ ;
5.  $\zeta(\tau_k) \notin A$  for any natural  $k$ ,  $i < k < j$ .



The length of a round is  $(j - i)\varepsilon$ .

**Lemma 4** *Let  $A = B_{\frac{\varepsilon}{3}}(Z^*)$  be a closed  $\frac{\varepsilon}{3}$ -neighbourhood of a limit point  $Z^*$  of the sequence  $\{\zeta(\tau_n)\}_{n=1}^\infty$ ; then, there are countably many rounds for  $A$ .*

*Proof* Indeed, there are countably many points from the set  $\{\zeta(\tau_n) \mid n \in \mathbb{N}\} \cap A$ . But there is no natural  $i$  such that both  $\zeta(\tau_i)$  and  $\zeta(\tau_{i+1})$  belong to  $A$  because

$$\rho(\zeta(\tau_i), \zeta(\tau_{i+1})) \geq d(\mathcal{L}(\tau_i), \mathcal{L}(\tau_{i+1})) = \varepsilon > \text{diam}(A). \quad (2)$$

Then, each point like  $\zeta(\tau_k) \in A$  is the start of the corresponding round and the end of previous round at the same time. So, we have countably many rounds for this  $A$ .

**Proposition 3** *There exists a nonempty set  $S \subset K^2$  ( $\text{diam}(S) < \varepsilon$ ) and a natural number  $m \geq 2$  such that there is a sequence of rounds for the set  $S$  that have the same-length  $m\varepsilon$ .*

*Proof* Let  $F$  be a limit point of the sequence  $\{\zeta(\tau_n)\}_{n=1}^\infty$ . Consider the set of all rounds for the closed ball  $B_{\frac{\varepsilon}{3}}(F)$ .

If we find countably many rounds such that their lengths are uniformly bounded from above, then we can find desired same-length rounds. Suppose the contrary.

In this case, we have a sequence of rounds for  $B_{\frac{\varepsilon}{3}}(F)$  the lengths of which grow infinitely. We shall show that there exists another required set.

Here we introduce the term *cage*. Consider a finite covering of  $K^2$  composed of  $B_{\frac{\varepsilon}{3}}(F)$  and other  $\frac{\varepsilon}{3}$ -balls. Let  $B_{\frac{\varepsilon}{3}}(F)$  be the ball number 1. Let us enumerate other balls from the covering as  $2, 3, \dots, N$ . *Cages* are defined as follows. Each point in  $K^2$  gets a number that is the minimal number among the numbers of the balls from the covering containing this point. Let  $i^{\text{th}}$  cage be the set of all points that marked with the number  $i$ ; this set may be empty. Note that we obtain  $N$  cages and the first one is the same as  $B_{\frac{\varepsilon}{3}}(F)$ .

Let us regard each round  $\zeta|_{[\tau_a, \tau_b]}$  for  $B_{\frac{\varepsilon}{3}}(F)$  as a tuple  $s_a, \dots, s_b$ , where  $s_i \in \{1, 2, \dots, N\}$  is the number of the cage that contains the point  $\zeta(\tau_i)$ . Note that  $s_a = s_b = 1$  and  $s_i \neq 1$  for all  $i$ ,  $a < i < b$ . Moreover, by (2), two neighbours  $\tau_i$  and  $\tau_{i+1}$  belong to different cages; so, in such a tuple all neighbour symbols should be different.

Since we have at most  $N$  cages, it follows that in each tuple there exist two equal numbers among the first  $N + 1$  symbols; if the length of a tuple is less than  $N + 1$ , the first and the last elements of this tuple, for instance, are such equal numbers. Due to the paragraph above, these equal numbers can not be neighbours in a tuple. Thus, we can consider ‘subtuples’ between the nearest equal numbers instead of the whole tuples. We get that the lengths of these subtuples are more than 2 and less than  $N + 2$ . Since we have countably many subtuples with uniformly bounded lengths, we can find countably many subtuples with the same-length. In the same way each subtuple starts with a symbol from  $\{1, 2, \dots, N\}$ . So we can find countably many subtuples starting

with the same number, say number  $k$ . It means that the corresponding curves start and finish in the  $k^{th}$  cage. Thus, we have countably many same-length restrictions of  $\zeta(\cdot)$ . Note that they are rounds for the  $k^{th}$  cage. Recall that the diameter of  $k^{th}$  cage is not more than  $\varepsilon$ .

Thus, we obtain countably many same-length rounds for the required set.

#### 5.4 Limit Curve

Let  $\{\zeta|_{[\tau_{i_n}, \tau_{i_n} + m\varepsilon]}\}_{n=1}^\infty$  be a sequence of rounds for a set  $S$  from Proposition 3. Consider the sequence of  $\zeta^n(\cdot) = (\zeta_L^n(\cdot), \zeta_M^n(\cdot)) : [0, m\varepsilon] \rightarrow K^2$  such that

$$\zeta^n(t) = \zeta(\tau_{i_n} + t) \quad \forall t \in [0, m\varepsilon].$$

Since  $\zeta_L^n(\cdot)$  and  $\zeta_M^n(\cdot)$  are 1-Lipschitz curves, thanks to Arzela–Ascoli theorem, this sequence of  $\zeta^n$  has a subsequence such that the corresponding subsequences of  $\zeta_L^n(\cdot)$  and  $\zeta_M^n(\cdot)$  converge to continuous maps  $\zeta_L^*(\cdot)$  and  $\zeta_M^*(\cdot)$ . Set  $\zeta^*(\cdot) = (\zeta_L^*(\cdot), \zeta_M^*(\cdot))$ .

**Proposition 4** *The following statements hold:*

1.  $\zeta^*(\cdot)$  is a good curve;
2.  $d(\zeta_L^*(0), \zeta_L^*(m\varepsilon)) < \varepsilon$ ;
3.  $d(\zeta_L^*(0), \zeta_M^*(0)) = d(\zeta_L^*(m\varepsilon), \zeta_M^*(m\varepsilon))$ .

*Proof* 1. The first statement follows from Lemma 3.

2. Since all  $\zeta_n(0)$  and  $\zeta_n(m\varepsilon)$  belong to  $S$ , it follows that  $\zeta^*(0)$  and  $\zeta^*(m\varepsilon)$  belong to the closure of this set. Hence,

$$d(\zeta_L^*(0), \zeta_L^*(m\varepsilon)) \leq \rho(\zeta^*(0), \zeta^*(m\varepsilon)) \leq \text{diam}(S) < \varepsilon.$$

3. We know that  $d(\mathcal{L}(\tau_k), \mathcal{M}(\tau_k)) - d(\mathcal{L}(\tau_{k+m}), \mathcal{M}(\tau_{k+m})) \rightarrow 0$  as  $k \rightarrow \infty$  because the distance between the players does not increase and is positive. Then,  $d(\zeta_L^n(0), \zeta_M^n(0)) - d(\zeta_L^n(m\varepsilon), \zeta_M^n(m\varepsilon)) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $d(\zeta_L^*(0), \zeta_M^*(0)) = d(\zeta_L^*(m\varepsilon), \zeta_M^*(m\varepsilon))$ .

#### 5.5 The Last Component

By this section we only need to show the contradiction to the original assumption. Recall that we assumed that there exists a 1-Lipschitz continuous curve  $\mathcal{M}(\cdot)$  such that if Lion follows the simple pursuit strategy then Man will never be captured.

From Proposition 2 and the last statement of Proposition 4, the image of  $[0, m\varepsilon]$  under  $\zeta_L^*(\cdot)$  coincides with the geodesic segment  $[\zeta_L^*(0), \zeta_L^*(m\varepsilon)]$ . Then, by Remark 2,

$$\begin{aligned} d(\zeta_L^*(0), \zeta_L^*(m\varepsilon)) &= d(\zeta_L^*(0), \zeta_L^*(\varepsilon)) + \dots + d(\zeta_L^*((m-1)\varepsilon), \zeta_L^*(m\varepsilon)) \\ &= m\varepsilon \geq 2\varepsilon. \end{aligned}$$

But it contradicts with the Statement 2 of Proposition 4. Thus, the assumption is false. In other words, at some time the distance between players will be less than  $\varepsilon$  and the capture will take place.

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